Problems Relating to a Thin Plane Annulus.

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CONTENTS.

	PAGE
§ 1. Introductory	195
§ 2. The elliptic anchor ring, and the annulus opening in an infinite wall	195
§ 3. The electrified thin annulus—surface density and charge	201
§ 4. Capacity of a thin annulus	202
§ 5. Axial motion of an annulus through infinite liquid	207

§ 1. *Introductory.*

The problems which arise in electrostatics and hydrodynamics, in regard to the thin annulus cut from a plane sheet of metal, are of some importance, more especially in regard to electrical instruments of precision, such as the electrometer. Their mathematical solution in an exact form is a matter of extreme difficulty, and only first approximations, which can be derived by simple methods, appear to have been used hitherto. In the present paper, higher approximations are obtained, to an order which appears to be effective for most of the applications which are of real importance. It is shown that the actual difference of radii of the circles bounding the annulus is of comparatively small significance in such magnitudes as the electrical capacity of the annulus, a result which could not readily be foreseen. The whole investigation is only carried to the second order of significance, but by a method—treating the annulus as a special case of the elliptic anchor ring—which can readily be extended to any desired order. The convergence of such approximate solutions is not discussed, but it is clearly analogous to the remarkable degree of convergence found by Lord Rayleigh in certain solutions of problems of vibration of discs in which eccentricity is taken into account.

§ 2. *The Elliptic Anchor Ring, and the Annulus Opening in an Infinite Wall.*

When electricity is freely distributed on a thin plane annulus of inner and outer radii (r_1, r_2), or otherwise, when a thin annulus of this form moves through infinite liquid, or serves as an opening in an infinite plane screen in liquid, we have at present no knowledge of the distribution of electricity, or of fluid motion. In the present paper, by taking such an annulus as a special case of an elliptic anchor ring, some progress is found possible with problems of this nature, though the investigation does not indicate a general exact solution. From a physical standpoint, however, the solutions obtained would appear to be sufficient.

If co-ordinates (μ, ζ, ω) in space are defined by

$$z = a\mu\zeta, \qquad x = c + a\sqrt{\{(1-\mu^2)(1+\zeta^2)\}}\cos\omega,$$
$$y = c + a\sqrt{\{(1-\mu^2)(1+\zeta^2)\}}\sin\omega,$$

the surfaces $\omega =$ constant are planes $y/x =$ constant, passing through the axis of z.

When ζ is constant, we have

$$\frac{z^2}{a^2\zeta^2} + \frac{(\rho-c)^2}{a^2(1+\zeta^2)} = 1, \qquad \rho = \sqrt{(x^2+y^2)},$$

which is an elliptic anchor ring, the generating ellipses of all such rings being confocal, and the rings becoming, when $c = 0$, oblate spheroids. This system of rings includes, as a special case, the plane annulus

$$z = 0, \qquad \rho = c \pm a,$$

if $c > a$. This corresponds to $\zeta = 0$.

It is seen without difficulty that the surfaces $(\zeta, \mu, \omega) =$ constant are a triply orthogonal system, so that (ζ, μ, ω) may be regarded as orthogonal co-ordinates of any point in space.

We find for the three corresponding space-elements,

$$\delta s_\mu = a\sqrt{\left\{\frac{\mu^2+\zeta^2}{1-\mu^2}\right\}}\,\delta\mu, \qquad \delta s_\zeta = a\sqrt{\left\{\frac{\mu^2+\zeta^2}{\zeta^2+1}\right\}}\,\delta\zeta,$$
$$\delta s_\omega = [c + a\sqrt{\{(1-\mu^2)(1+\zeta^2)\}}]\,\delta\omega,$$

resembling the case of oblate spheroidal co-ordinates in space. The internal and external radii of the special annulus are a and c.

We find also for Laplace's equation, when there is symmetry about the axis z

$$\left(\sqrt{(1+\zeta^2)}\frac{\partial}{\partial\zeta}\right)^2\phi + \left(\sqrt{(1-\mu^2)}\frac{\partial}{\partial\mu}\right)^2\phi$$
$$= -\frac{a}{c}\sqrt{(1+\zeta^2, 1-\mu^2)}\left\{\frac{\partial}{\partial\zeta}\left(1+\zeta^2\cdot\frac{\partial\phi}{\partial\zeta}\right) + \frac{\partial}{\partial\mu}\left(1-\mu^2\frac{\partial\phi}{\partial\mu}\right)\right\}$$

which obviously has no exact solutions which are functions of ζ or μ only.

Taking more convenient variables α and β, where

$$\zeta = \sinh\alpha, \qquad \mu = \sin\beta,$$

$$\frac{\partial^2\phi}{\partial\alpha^2} + \frac{\partial^2\phi}{\partial\beta^2} = -\frac{a}{c}\left\{\cos\beta\frac{\partial}{\partial\alpha}\left(\cosh\alpha\frac{\partial\phi}{\partial\alpha}\right) + \cosh\alpha\frac{\partial}{\partial\beta}\left(\cos\beta\frac{\partial\phi}{\partial\beta}\right)\right\}$$

and if a/c is small we may solve this in powers of a/c by successive approximations, the term independent of a/c in any solution being a solution of

$$\frac{\partial^2\phi}{\partial\alpha^2} + \frac{\partial^2\phi}{\partial\beta^2} = 0,$$

$\alpha = \text{const.}$ defines the elliptic, and $\beta = \text{const.}$ the hyperbolic, anchor rings which belong to the orthogonal system.

On the annulus $z = 0$, ρ ranging from $a-c$ to $a+c$—this is a member of the system—we have $\zeta = 0$, and, therefore, $\alpha = 0$. On the rest of the plane $z = 0$, we have $\mu = 0$, and, therefore, $\beta = 0$. Moreover,

$$\left(\frac{\partial\phi}{\partial\zeta}\right)_{\alpha=0} = \left(\frac{\partial\phi}{\partial\alpha}\cdot\frac{\partial\alpha}{\partial s_\zeta}\right)_{\alpha=0} = \left(\frac{\partial\phi}{\partial\alpha}\cdot\frac{\partial\alpha}{\partial\zeta}\cdot\frac{\partial\zeta}{\partial s_\zeta}\right)_{\alpha=0}$$

$$= \left(\frac{\partial\phi}{\partial\alpha}\cdot\frac{1}{\cosh\alpha}\cdot a^{-1}\sqrt{\frac{\zeta^2+1}{\mu^2+\zeta^2}}\right)_{\substack{\alpha=0\\ \zeta=0}} = \frac{1}{a\mu}\left(\frac{\partial\phi}{\partial\alpha}\right)_0,$$

$$\left(\frac{\partial\phi}{\partial s_\mu}\right)_{\beta=0} = \left(\frac{\partial\phi}{\partial\beta}\cdot\frac{\partial\beta}{\partial\mu}\cdot\frac{\partial\mu}{\partial s_\mu}\right)_0 = \left[\frac{\partial\phi}{\partial\beta}\cdot\frac{1}{\cos\beta}\cdot\frac{1}{a}\sqrt{\frac{1-\mu^2}{\mu^2+\zeta^2}}\right]_0 = \frac{1}{a\zeta}\left(\frac{\partial\phi}{\partial\beta}\right)_0,$$

and when $\zeta = 0$, on the annulus,

$$\rho - c = a\sqrt{(1-\mu^2)},$$

or

$$\mu = \sqrt{\left\{1-\frac{(\rho-c)^2}{a^2}\right\}}, \qquad a\mu = \sqrt{\{(\overline{a+c}-\rho)(\overline{a-c}+\rho)\}}$$

making β and μ vanish at each edge.

On the rest of the plane, when $\mu = 0$,

$$a^2(1+\zeta^2) = (\rho-c)^2, \quad \zeta = \sqrt{\left\{\frac{(\rho-c)^2}{a^2}-1\right\}}, \quad a\zeta = \sqrt{(\rho-\overline{c+a}\,.\,\rho-\overline{c-a})},$$

so that ζ and α also vanish at each edge.

When a is sufficiently small in comparison with c, the solutions approximate to those of

$$\frac{\partial^2\phi}{\partial\alpha^2}+\frac{\partial^2\phi}{\partial\beta^2} = 0,$$

of which the only solutions not dependent on β are

$$\phi = \mathrm{A}+\mathrm{B}\alpha.$$

Consider the solution $\phi = \alpha$. This makes ϕ zero when $\alpha = 0$, and when $\beta = 0$, on the plane $z = 0$,

$$\frac{\partial\phi}{\partial n} = \frac{1}{a\zeta}\frac{\partial\phi}{\partial\beta} = 0.$$

It is thus apparently suitable for the motion of a liquid through a thin annular opening in the barrier $z = 0$, satisfying all the conditions approximately. These are (1) the differential equation; (2) ϕ constant when $\alpha = 0$; (3) $\partial\phi/\partial n = 0$ when $\beta = 0$; (4) ϕ continuous at the edges where $\alpha = 0$.

We find also on the aperture,

$$\frac{\partial\phi}{\partial n} = \frac{1}{a\mu}\left(\frac{\partial\phi}{\partial\alpha}\right)_0 = \frac{1}{a\mu} = \frac{1}{\sqrt{\{(a+c-\rho)(a-c+\rho)\}}}.$$

The velocity is infinite to order $\frac{1}{2}$ at both edges. This is to be expected, and might in fact have been written down at once as the necessary approximate solution, from our knowledge of the effect at the edge in the case of a circular aperture, or a stream flowing past a circular obstacle. We must now proceed by successive approximations to a solution of the form

$$\phi = \alpha + \frac{a}{c} f_1 + \left(\frac{a^2}{c^2}\right) f_2 + \left(\frac{a^3}{c^3}\right) f_3 + \ldots$$

where the functions f_n must possess the following properties:—

(1) ϕ is a solution of the differential equation, to some specified order of a/c.

(2) $f_n = 0$ when $\alpha = 0$.

(3) $\partial f_n / \partial \beta = 0$ when $\beta = 0$.

The second secures that $\phi = 0$ over the aperture, and the third that $\partial\phi/\partial z = 0$ over the rest of the plane $z = 0$. The third is satisfied necessarily if f_n contains only cosines of β and its multiples as multipliers of its terms in α. It is, of course, understood that the series may cease to be applicable when α is large, and require another form of continuation.

Writing $\phi = \alpha + \frac{a}{c}\, \psi_1$ in the differential equation, we find, neglecting terms of order $(a/c)^2$,

$$\frac{\partial^2 \psi_1}{\partial \alpha^2} + \frac{\partial^2 \psi_1}{\partial \beta^2} = -\left\{ \cos\beta \frac{\partial}{\partial \alpha} \cosh\alpha \frac{\partial}{\partial \alpha} + \cosh\alpha \frac{\partial}{\partial \beta} \cos\beta \frac{\partial \alpha}{\partial \beta} \right\} = -\cos\beta \sinh\alpha,$$

of which the particular integral is

$$\psi_1 = -\tfrac{1}{2}\alpha \cosh\alpha \cos\beta.$$

No complementary function satisfying

$$\frac{\partial^2 \psi_1}{\partial \alpha^2} + \frac{\partial^2 \psi_1}{\partial \beta^2} = 0$$

must be added, for it would represent *another* approximate solution of the original equation, multiplied by c/a, in its first term. We must, in fact, throughout the work, notice that no term appears which is itself a solution of this equation, but this will not occur if the particular integrals are obtained according to the ordinary rules in linear differential equations.

The second approximation to ϕ is therefore

$$\phi = \alpha - \frac{\alpha}{2} \frac{a}{c} \cosh\alpha \cos\beta$$

where $$f_1 = -\tfrac{1}{2}\alpha \cosh\alpha \cos\beta$$

with the necessary properties, as a function of α or of β.

Now write

$$\phi = \alpha - \frac{a\alpha}{2c}\cosh\alpha\cos\beta + \frac{a^2}{c^2}\psi_2$$

and neglecting a^3/c^3, we have

$$\frac{\partial^2\psi_2}{\partial\alpha^2} + \frac{\partial^2\psi_2}{\partial\beta^2} = +\left(\cos\beta\frac{\partial}{\partial\alpha}\cosh\alpha\frac{\partial}{\partial\alpha} + \cosh\alpha\frac{\partial}{\partial\beta}\cos\beta\frac{\partial}{\partial\beta}\right)\frac{\alpha}{2}\cosh\alpha\cos\beta$$

$$= \tfrac{1}{8}(3\sinh 2\alpha + 2\alpha\cosh 2\alpha) + \tfrac{1}{8}\cos 2\beta\,(3\sinh 2\alpha - 2\alpha),$$

and on solution

$$\psi_2 = \tfrac{1}{32}(\sinh 2\alpha + 2\alpha\cosh 2\alpha) + \tfrac{1}{32}\cos 2\beta\,(2\alpha + 3\alpha\cosh 2\alpha).$$

This also has the properties $(\psi_2)_{\alpha=0} = 0$, $(\partial\psi_2/\partial\beta)_{\beta=0} = 0$.

The third approximation to ϕ is now known. If ψ_3 is the next function required,

$$\frac{\partial^2\psi_3}{\partial\alpha^2} + \frac{\partial^2\psi_3}{\partial\beta^2} = -\left\{\cos\beta\frac{\partial}{\partial\alpha}\cosh\alpha\frac{\partial}{\partial\alpha} + \cosh\alpha\frac{\partial}{\partial\beta}\cos\beta\frac{\partial}{\partial\beta}\right\}\psi_2,$$

from which ψ_3 is of the form $\mathrm{P}\cos\beta + \mathrm{Q}\cos 3\beta$, where P and Q are functions of α vanishing when α is zero. The comparative simplicity of the coefficients in these functions does not, however, continue, although it is quite easy to continue the approximation to a high order. It has already gone far enough for most purposes, on account of the great initial convergency of the series over the aperture. We may, therefore, write, to order a^2/c^2,

$$\phi = \alpha - \frac{a\alpha}{2c}\cosh\alpha\cos\beta + \frac{a^2}{32c^2}(\sinh 2\alpha + 2\alpha\cosh 2\alpha)$$
$$+ \frac{a^2\cos 2\beta}{32c^2}(2\alpha + 3\alpha\cosh 2\alpha).$$

The normal velocity over the aperture is therefore

$$\frac{1}{a\mu}\left(\frac{\partial\phi}{\partial\alpha}\right)_{\alpha=0} = \frac{1}{a\mu}\left\{1 - \frac{a}{2c}\cos\beta + \frac{a^2}{8c^2} + \frac{5}{32}\frac{a^2}{c^2}\cos 2\beta\right\}$$

where $(\rho - c)/a = \sqrt{(1-\mu^2)} = \cos\beta$,

and the velocity thus becomes

$$(c+a-\rho)^{-\frac{1}{2}}(\rho-c+a)^{-\frac{1}{2}}\left\{1 - \frac{a^2}{32c^2} - \frac{\rho-c}{2c} + \frac{5}{16}\frac{(\rho-c)^2}{c^2}\right\}$$

the value at the centre being

$$\frac{1}{a}\left(1 - \frac{a^2}{32c^2}\right).$$

For a velocity U at the central radius $\rho = c$, the velocity U_ρ at ρ is

$$\mathrm{U}_\rho = \frac{a\mathrm{U}}{\sqrt{\{(c+a-\rho)(\rho-c+a)\}}}\left\{1 - \frac{\rho-c}{2c} + \frac{5}{16}\frac{(\rho-c)^2}{c^2}\right\}.$$

The volume V discharged through the aperture in unit time is

$$V = \int_{c-a}^{c+a} 2\pi\rho U_\rho d\rho = 4\pi a U \int_0^{\pi/2} d\phi \left\{\rho - \frac{\rho(\rho-c)}{2c} + \frac{5}{16}\frac{\rho(\rho-c)^2}{c^2}\right\}$$

(with $\rho = \overline{c-a} \,.\, \cos^2\phi + \overline{c+a} \,.\, \sin^2\phi$),

and after an easy integration,

$$V = 2\pi^2 ac U \left(1 - \frac{3a^2}{32c^2}\right)$$

no term appearing on a/c. The average velocity over the area is

$$\frac{\pi}{2} U \left(1 - \frac{3a^2}{32c^2}\right)$$

where U is the velocity at the central radius.

The stream surfaces are almost identical with the hyperbolic anchor rings

$$\frac{(\rho-c)^2}{a^2(1-\mu^2)} - \frac{z^2}{a^2\mu^2} = 1$$

where μ is constant.

It is evident that the formulæ, even to this approximation, may be used with accuracy of a few per cent. even in such a case as $a = \frac{1}{2}c$.

The Bessel-Fourier solution to this order is, in the plane of the annulus,

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=0} = \int_0^\infty \lambda J_0(\lambda\rho)\, d\lambda \int_{c-a}^{c+a} \mu J_0(\lambda\mu)\, U_\mu\, d\mu$$

so that more generally

$$\phi = -\int_0^\infty J_0(\lambda\rho)\, e^{-\lambda z}\, d\lambda \int_{c-a}^{c+a} \mu\, J_0(\lambda\mu)\, U_\mu\, d\mu$$

and on the axis of z, where $\rho = 0$,

$$\phi = -\int_{c-a}^{c+a} \mu\, U_\mu \frac{d\mu}{\sqrt{(z^2+\mu^2)}}.$$

If z is less than $c-a$, we can obtain a convergent series in powers of z, of the form

$$\phi = -\sum_0^\infty (-)^n \frac{2n\,!}{2^{2n}(n\,!)^2} \,.\, z^{2n} \int_{c-a}^{c+a} \frac{d\mu\, U_\mu}{\mu^{2n}}$$

and its appropriate generalisation, in the usual way, as a series of zonal harmonics, valid when $r < c-a$,

$$\phi = -Ua \sum_0^\infty (-)^n \frac{2n\,!}{2^{2n}(n\,!)^2} r^{2n} P_{2n}(\mu) \int_{c-a}^{c+a} \frac{d\mu\,\{1-(\mu-c)/2c+5(\mu-c)^2/16c^2\}}{\mu^{2n}\sqrt{\{(c+a-\mu)(\mu-c+a)\}}}.$$

The corresponding harmonic series for a distant point can obviously be written down in a similar manner.

As a special case, the value of ϕ over the plane $z = 0$, for points at which $\rho < c-a$, on part of the barrier, is

$$\phi = -\sum_0^\infty \left[(-)^n \frac{2n\,!}{2^{2n}(n\,!)^2}\right]^2 \rho^{2n} \int_{c-a}^{c+a} \frac{d\mu\, \mathrm{U}_\mu}{\mu^{2n}}$$

$$= -\mathrm{U}a \int_{c-a}^{c+a} \mathrm{K}\left(\frac{\rho}{\mu}\right) \frac{d\mu}{\sqrt{\{(c+a-\mu)(\mu-c+a)\}}} \left\{1-\frac{\mu-c}{2c}+\frac{5}{16}\frac{(\mu-c)^2}{c^2}\right\}$$

where $\mathrm{K}(\rho/\mu)$ is the complete elliptic integral of modulus ρ/μ. Since ρ and μ are nearly equal, this can be evaluated approximately by use of the logarithmic form of K.

§ 3. *The Electrified Thin Annulus. Surface Density and Charge.*

The last problem is analytically identical with that of the distribution of free electricity on a thin annular conductor, and we can replace normal velocity over the aperture by surface density over the annular conductor. For V, the electric potential, is constant over the conductor, and $\partial \mathrm{V}/\partial z = 0$ over the rest of its plane. If c/a is very large—the radii being $c-a$, $c+a$—the potential at a point close to the conductor is evidently that produced by an indefinitely long thin plate of breadth $2a$, which is a special case of the elliptic cylinder, when the minor axis of the principal elliptic section degenerates to zero. If the same axes of this section were, in the first place, $a\zeta_0$, $a\sqrt{(1+\zeta_0^2)}$, the potential at an external point (z, ρ) would be, as regards its variable part, proportional to

$$\int_\lambda \frac{dt}{\sqrt{\{(a^2\zeta_0^2+t)\}\{(a^2(1+\zeta_0^2)+t)\}}}.$$

Writing

$$t = a^2\zeta_0^2 \sinh^2 u - a^2(1+\zeta_0^2)\cosh^2 u,$$

the integral becomes

$$2\int_\lambda du = (2u)_{t=\lambda} + \text{const.}$$

Now λ is the parameter of the ellipse through (z, ρ) defined by

$$\frac{z^2}{a^2\zeta_0^2+\lambda} + \frac{\rho^2}{a^2+a^2\zeta_0^2+\lambda} = 1.$$

With the spheroidal co-ordinates $z = a\mu\zeta$, $\rho = a\sqrt{(1-\mu^2 \cdot 1+\zeta^2)}$ we find $\lambda = a^2(\zeta^2-\zeta_0^2)$, so that $(u)_{t=\lambda}$ is given by

$$a^2\zeta_0^2 \sinh^2 u - a^2(1+\zeta_0^2)\cosh^2 u = \lambda = a^2(\zeta^2-\zeta_0^2),$$

or

$$(u)_{t=\lambda} = \sinh^{-1}\zeta = \alpha,$$

where α is the variable occurring in the previous section.

Thus, when c/a is large, the potential of the elliptic anchor ring of oblate type is of the form $\mathrm{A}+\mathrm{B}\alpha$ at external points just outside, if it is very flat.

This justifies our selection of α as the solution from which a further approximation was to be obtained, and rules out other functions, which at first sight might appear suitable.

The surface density of a free distribution on a thin annulus of radii $c \pm a$ is therefore proportional to the value of $\partial\phi/\partial n$ in the last section, or

$$\sigma = \frac{\sigma_0 a}{\sqrt{\{(c+a-\rho)(\rho-c+a)\}}}\left\{1-\frac{\rho-c}{2c}+\frac{5}{16}\frac{(\rho-c)^2}{c^2}\right\}$$

and the total charge Q on one side becomes on integration

$$Q = 2\pi^2 ac\sigma_0\left(1-\frac{3a^2}{32c^2}\right),$$

σ_0 being the density at the central circle.

In order to find the capacity of the annulus, we require the potential produced by this distribution at a point of the annulus. Hitherto, it has been an additive constant of unknown magnitude in the investigation. We find it by direct integration for a point $\rho = c+a$ on the outer edge, for it is the same at all points on the annulus.

4. *Capacity of a Thin Annulus.*

The potential of a circular ring of charge E at a point distant f from its centre,—f not less than the radius,—in its plane is

$$V = \frac{E}{\pi}\int_0^\pi \frac{d\theta}{\sqrt{(\rho^2+f^2-2f\rho\cos\theta)}}$$

where ρ is the radius of the ring. This becomes, if K is the complete elliptic function,

$$V = \frac{2E}{\pi}\cdot\frac{1}{\rho+f}K\left(\frac{2\sqrt{(\rho f)}}{\rho+f}\right).$$

Let $f = c+a$, and let ρ be approximately equal to f. The modulus k of the elliptic function is $2\sqrt{(\rho f)}/(\rho+f)$, so that, in the usual notation, $k' = (f-\rho)/(f+\rho)$ and is nearly zero. Thus, in fact, on expansion in powers of a/c or $(\rho-f)/c$,

$$k' = \frac{a+c-\rho}{a+c+\rho} = \frac{a+c-\rho}{2c+(a+\rho-c)}$$

$$= \frac{a+c-\rho}{2c}\left\{1-\frac{a+\rho-c}{2c}+\frac{(a+\rho-c)^2}{4c^2}\right\}$$

to a sufficient order. Moreover*

$$K(k) = \left(1+\tfrac{1}{4}k'^2+\frac{9}{64}k'^4\right)\log_e\left(\frac{4}{k'}\right)-2\left(\frac{1}{1\,.\,2}\right)\tfrac{1}{4}k'^2-2\left(\frac{1}{1\,.\,2}+\frac{1}{3\,.\,4}\right)\frac{9}{64}k'^4$$

where the approximation

$$K(k) = (1+\tfrac{1}{4}k'^2)\log_e\left(\frac{4}{k'}\right)-\tfrac{1}{4}k'^2$$

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* Cayley, 'Elliptic Functions,' p. 541.

is all that we require, to relative order a^2/c^2. Thus

$$\mathrm{K}(k) = -\left\{1+\frac{(a+c-\rho)^2}{16c^2}\right\}\log\left(\frac{a+c-\rho}{8c}\right)$$

$$+\frac{a+\rho-c}{2c}\left(1-\frac{a+\rho-c}{4c}\right)\left(1+\frac{(a+c-\rho)^2}{16c^2}\right)\frac{(a+c-\rho)^2}{16c^2}$$

$$= -\left(1+\frac{(a+c-\rho)^2}{16c^2}\right)\log\left(\frac{a+c-\rho}{8c}\right)$$

$$+\frac{a+\rho-c}{2c}-\frac{3a^2+3(\rho-c)^2+2a(\rho-c)}{16c^2},$$

also

$$\frac{1}{\rho+f}=\frac{1}{\rho+a+c}=\frac{1}{2c}\left\{1-\frac{a+\rho-c}{2c}+\frac{(a+\rho-c)^2}{4c^2}\right\},$$

and we find ultimately

$$\frac{2c\pi\mathrm{V}}{2\mathrm{E}} = -\left(1-\frac{a+\rho-c}{2c}+\frac{4(a+\rho-c)^2+(a+c-\rho)^2}{16c^2}\right)\log_e\left(\frac{a+c-\rho}{8c}\right)$$

$$+\frac{a+\rho-c}{2c}-\frac{7a^2+7(\rho-c)^2+10a(\rho-c)}{16c^2}$$

Writing now, $\mathrm{E} = 2\pi\rho\, d\rho\,.\,\sigma$, where σ is the surface density of the last section, the potential of the annulus is obtained in the form

$$\mathrm{V}_0 = \frac{2}{c}\int_{c-a}^{c+a}\rho\sigma\, d\rho\,.\,f(\rho)$$

where $f(\rho)$ is the function of ρ on the right of the previous equation, and

$$\sigma = \frac{\sigma_0 a}{\sqrt{\{(c+a-\rho)(\rho-c+a)\}}}\left\{1-\frac{\rho-c}{2c}+\frac{5}{16}\frac{(\rho-c)^2}{c^2}\right\}.$$

The integration is very tedious. With $\rho = (c-a)\cos^2\theta+(c+a)\sin^2\theta$, we find

$$\mathrm{V}_0 = \frac{4\theta\sigma_0}{c}\int_0^{\pi/2}\rho\, d\theta\,.\,\mathrm{F}(\rho)$$

where

$$\mathrm{F}(\rho) = \mathrm{F}_1(\rho)+\mathrm{F}_2(\rho),$$

$$\mathrm{F}_1(\rho) = -\left\{1-\frac{a+2\rho-2c}{2c}+\frac{5a^2+14(\rho-c)^2+10a(\rho-c)}{16c^2}\right\}\log\left(\frac{a+c-\rho}{8c}\right),$$

$$\mathrm{F}_2(\rho) = \frac{a+\rho-c}{2c}-\frac{7a^2+11(\rho-c)^2+14a(\rho-c)}{16c^2}.$$

If
$$c\mathrm{V}_0 = 4a\sigma_0(\mathrm{V}_0{}^1+\mathrm{V}_0{}^2)$$

where $\mathrm{V}_0{}^1$, $\mathrm{V}_0{}^2$ are the portions of the integral arising from $\mathrm{F}_1(\rho)$, $\mathrm{F}_2(\rho)$, we easily find

$$\mathrm{V}_0{}^2 = \frac{\pi a(a+2c)}{8c}\quad\frac{7\pi a^2}{32c}-\frac{11\pi a^2}{64c}-\frac{7\pi a^3}{32c^2} = \frac{\pi a}{4}\left(1-\frac{17a}{16c}\right)$$

to the present order. Moreover

$$\log\left(\frac{a+c-\rho}{8c}\right) = \log\frac{a}{4c}+2\log\cos\theta$$

and $$\mathrm{V}_0{}^1 = -\log\frac{a}{4c}\int_0^{\pi/2}\rho\,d\theta\left\{1-\frac{a+2\rho-2e}{2c}+\frac{5a^2+14(\rho-c)^2+10a(\rho-c)}{16c^2}\right\}$$

$$-2\int_0^{\pi/2}\left(1-\frac{a}{2c}+\frac{5a^2}{16c^2}\right)\log\cos\theta\,d\theta\,\{(c-a)\cos^2\theta+(c+a)\sin^2\theta\}$$

$$+\frac{4a}{2c}\left(1-\frac{5a}{8c}\right)\int_0^{\pi/2}[(c-a)\cos^2\theta+(c+a)\sin^2\theta]$$
$$\log\cos\theta\,d\theta\,(2\sin^2\theta-1)$$

$$-\frac{7a^2}{4c^2}\int_0^{\pi/2}[(c-a)\cos^2\theta+(c+a)\sin^2\theta]\log\cos\theta\,d\theta\,(2\sin^2\theta-1)^2.$$

These integrals can all be evaluated. If they are called I_1, I_2, I_3, I_4, then

$$\mathrm{V}_0{}^1 = \log\frac{4c}{a}\,.\,\mathrm{I}_1-2\mathrm{I}_2+\frac{2a}{c}\left(1-\frac{5a}{8c}\right)\mathrm{I}_3-\frac{7a^2}{4c^2}\mathrm{I}_4\,.$$

We find

$$\mathrm{I}_1 = \frac{\pi c}{2}\left(1-\frac{a}{2c}+\frac{5a^2}{16c^2}\right)-\frac{\pi a^2}{4c}\left(1-\frac{5a}{8c}\right)+\frac{7\pi a^2}{32c}$$

$$= \frac{\pi c}{2}\left(1-\frac{a}{2c}+\frac{a^2}{4c^2}\right)$$

to the requisite order.

It is well known, and easily proved, that

$$\int_0^{\pi/2}\log\cos\theta\,d\theta = -\frac{\pi}{2}\log 2.$$

By an easy integration by parts, we find also

$$2n\int_0^{\pi/2}\sin^{2n}\theta\log\cos\theta\,d\theta = (2n-1)\int_0^{\pi/2}\sin^{2n-2}\theta\log\cos\theta\,d\theta-\int_0^{\pi/2}\sin^{2n}\theta\,d\theta,$$

so that

$$\int_0^{\pi/2}\sin^2\theta\log\cos\theta\,d\theta = -\frac{\pi}{4}\log 2-\frac{\pi}{4},$$

$$\int_0^{\pi/2}\sin^4\theta\log\cos\theta\,d\theta = -\frac{3\pi}{16}\log 2-\frac{15\pi}{64},$$

$$\int_0^{\pi/2}\sin^6\theta\log\cos\theta\,d\theta = -\frac{5\pi}{32}\log 2-\frac{85\pi}{384}.$$

Thus

$$\mathrm{I}_2 = \left(1-\frac{a}{c}+\frac{5a^2}{16c^2}\right)\int_0^{\pi/2}\{(c-a)+2a\sin^2\theta\}\log\cos\theta\,d\theta$$

$$= c\left(1-\frac{a}{c}+\frac{5a^2}{16c^2}\right)\left(-\frac{\pi}{2}\log 2\right)-\frac{\pi a}{2}\left(1-\frac{a}{c}+\frac{5a^2}{16c^2}\right)$$

$$= -\frac{\pi c}{2}\log 2\left(1-\frac{a}{c}+\frac{5a^2}{16c^2}\right)-\frac{\pi c}{2}\,.\,\frac{a}{c}\left(1-\frac{a}{c}\right)$$

to the proper order,

$$\mathrm{I}_3 = \int_0^{\pi/2} \{-(c-a)+(2c-4a)\sin^2\theta+4a\sin^4\theta\}\log\cos\theta\,d\theta$$

$$= \frac{\pi}{2}\log 2\,.\,(c-a)-(2c-4a)\left(\frac{\pi}{4}\log 2+\frac{\pi}{4}\right)-4a\left(\frac{3\pi}{16}\log 2+\frac{15\pi}{64}\right)$$

$$= -\frac{\pi c}{4}\log 2\,.\left(\frac{a}{c}\right)-\frac{\pi c}{2}\left(1-\frac{a}{8c}\right),$$

$$\mathrm{I}_4 = \int_0^{\pi/2} [c-a+(6a-4c)\sin^2\theta+(4c-12a)\sin^4\theta+8a\sin^6\theta]\log\cos\theta\,d\theta$$

$$= -\frac{\pi}{2}\log 2\,(c-a)+(4c-6a)\left(\frac{\pi}{4}\log 2+\frac{\pi}{4}\right)$$
$$+(12a-4c)\left(\frac{3\pi}{16}\log 2+\frac{15\pi}{64}\right)-8a\left(\frac{5\pi}{32}\log 2+\frac{85\pi}{384}\right)$$

$$= -\frac{\pi c}{4}\log 2+\frac{\pi c}{16}\left(1-\frac{22a}{3c}\right),$$

and accordingly

$$\mathrm{V}_0{}^1 = \log\frac{4c}{a}\,.\,\mathrm{I}_1-2\mathrm{I}_2+\frac{2a}{c}\left(1-\frac{5a}{8c}\right)\mathrm{I}_3-\frac{7a^2}{4c^2}\,\mathrm{I}_4$$

$$= \frac{\pi c}{2}\left(1-\frac{a}{2c}+\frac{a^2}{4c^2}\right)\log\frac{4c}{a}+\pi c\,.\,\frac{a}{c}\left(1-\frac{a}{c}\right)+\frac{2a^2}{c^2}\left(-\frac{\pi c}{4}\log 2\right)$$
$$-\pi c\,.\,\frac{a}{c}\left(1-\frac{5a}{8c}\right)\left(1-\frac{a}{8c}\right)+\frac{7\pi c}{16}\frac{a^2}{c^2}\log 2-\frac{7}{64}\pi c\,.\,\frac{a^2}{c^2}$$

$$= \frac{\pi c}{2}\left(1-\frac{a}{2c}+\frac{a^2}{4c^2}\right)\log\frac{4c}{a}+\pi c\log 2\left(1-\frac{a}{c}+\frac{a^2}{4c^2}\right)-\frac{23}{64}\pi c\,.\,\frac{a^2}{c^2}$$

$$= \frac{\pi c}{2}\left(1-\frac{a}{2c}+\frac{a^2}{4c^2}\right)\log\frac{16c}{a}-\pi c\,.\,\frac{a^2}{c^2}\cdot\frac{23}{64},$$

the terms involving the logarithms condensing elegantly.

The whole potential of the annulus is finally

$$\mathrm{V}_0 = \frac{4a\sigma_0}{c}(\mathrm{V}_0{}^1+\mathrm{V}_0{}^2)$$

$$= \frac{4a\sigma_0}{c}\cdot\frac{\pi c}{2}\left(1-\frac{a}{2c}+\frac{a^2}{4c^2}\right)\log\frac{16c}{a}+\frac{4a\sigma_0}{c}\left\{\frac{\pi c}{4}\cdot\frac{a}{c}-\pi c\left(\frac{17}{64}\frac{a^2}{c^2}+\frac{23}{64}\frac{a^2}{c^2}\right)\right\}$$

$$= 2\pi a\sigma_0\left\{\left(1-\frac{a}{2c}+\frac{a^2}{4c^2}\right)\log\frac{16c}{a}+\frac{a}{2c}\left(1-\frac{5a}{2c}\right)\right\}$$

where σ_0 is the central density.

The total charge on one side was

$$\mathrm{Q} = 2\pi^2 ac\sigma_0\left(1-\frac{3a^2}{32c^2}\right)$$

and the complete charge being twice this amount, the capacity of the annulus is

$$2\pi c\left(1-\frac{3a^2}{32c^2}\right)\Big/\left\{\left(1-\frac{a}{2c}+\frac{a^2}{4c^2}\right)\log\frac{16c}{a}+\frac{a}{2c}\left(1-\frac{5a}{2c}\right)\right\}$$

which can be used with some accuracy if a is less than about $\frac{1}{5}c$. A higher approximation could be worked out readily by this analysis.

Numerical calculations give results agreeing in general terms with those corresponding to the ordinary anchor ring of radii (a, c), for which the capacity is already known.*

The error in this formula is roughly of order $(a/c)^3$, or only 1 or 2 per cent. if $a = \frac{1}{4}c$. If a is so small as $\frac{1}{10}c$, we may drop the terms in $(a/c)^2$, and use the simpler form

$$2\pi c\left\{1+\frac{a}{2c}\left(\log_e\frac{16c}{a}-1\right)\right\}.$$

In the Table typical values are given. The radii are $c \pm a$, so that c is the mean radius of the annulus:—

a/c.	(Capacity)/$2\pi c$.
0·05	0·177
0·1	0·205
0·15	0·227
0·2	0·247
0·25	0·263
0·3	0·283

The presence of an inner edge much increases the capacity, as the laws of electrical distribution require that the surface density becomes large there. A complete circular disc of radius $2c$, and therefore of mean radius c, has a capacity

$$4c/\pi \text{ or } 2\pi c\times\frac{2}{\pi^2} = 2\pi c\times 0{\cdot}20264.$$

Thus, an annulus of the same *mean* radius, and $a/c = 1/10$, has practically the same capacity. There will be another annulus which has a *maximum* capacity for a given mean radius, but the present approximation does not allow us to determine it precisely. It is clearly one with a very small opening at the centre.

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* Hicks, 'Phil. Trans.,' and Dyson, 'Phil. Trans.,' 1893.

§ 5. *Axial Motion of an Annulus through Infinite Liquid.*

When an ellipsoid of semi-axes (a, b, c) moves parallel to the axis x, corresponding to a, it creates a velocity potential at (x, y, z) equal to

$$\phi = \frac{abc\mathrm{U}}{2-\alpha_0} \cdot x \int_\lambda^\infty \frac{dt}{(a^2+t)^{3/2}(b^2+t)^{1/2}(c^2+t)^{1/2}}$$

where

$$\alpha_0 = abc \int_0^\infty \frac{dt}{(a^2+t)^{3/2}(b^2+t)^{1/2}(c^2+t)^{1/2}}$$

U is the velocity of the ellipsoid, and λ is the parameter of the confocal through (x, y, z). For the special case of an elliptic cylinder, $c = \infty$, and we find

$$\alpha_0 = ab \int_0^\infty \frac{dt}{(a^2+t)^{3/2}(b^2+t)^{1/2}} = -\frac{2}{b^2-a^2}\Big(\coth u\Big)_{t=0}^{t=\infty}$$

where
$$t = -a^2\cosh^2 u + b^2\sinh^2 u,$$

or
$$\alpha_0 = -\frac{2ab}{b^2-a^2}\Big(\coth u\Big)_{u=\coth^{-1} b/c}^{u=\infty} = -\frac{2}{b^2-a^2}\left(\frac{b}{a}-1\right) = -\frac{2b}{(a+b)}.$$

We have assumed that b is the semi-axis *major*, so that it moves along the minor axis. The value of ϕ for the elliptic cylinder becomes

$$\phi = \frac{b\mathrm{U}x}{b-a}(\coth^{-1} u - 1) \text{ with } \lambda = -a^2\cosh^2 u + b^2\sinh^2 u,$$

or
$$\phi = \frac{b\mathrm{U}x}{b-a}\left\{\sqrt{\left(\frac{b^2+\lambda}{a^2+\lambda}\right)}-1\right\}.$$

This result is not usually given in the text-books, although very convenient. If we use instead the co-ordinates (μ, ζ) defined by

$$x = a\mu\zeta, \qquad y = a\sqrt{(1-\mu^2 \,.\, 1+\zeta^2)}$$

and define the moving cylinder by $\zeta = \zeta_0$, the equation of the cylinder is

$$\frac{x^2}{a^2\zeta_0^2} + \frac{y^2}{a^2(1+\zeta_0^2)} = 1,$$

and the parameter λ of an external point is

$$\lambda = a^2(\zeta^2 - \zeta_0^2)$$

while a^2 is replaced by $a^2\zeta_0^2$, and b^2 by $a^2(1+\zeta_0^2)$, so that

$$\phi = \frac{a\mathrm{U}\mu}{1-\zeta_0/\sqrt{(1+\zeta_0^2)}}\{\sqrt{(1+\zeta^2)}-\zeta\}.$$

This is the velocity potential created by the motion of the cylinder ζ_0 of the family

$$\frac{x^2}{a^2\zeta^2} + \frac{y^2}{a^2(1+\zeta^2)} = 1$$

parallel to the axis of x. With co-ordinates $\alpha = \sinh^{-1}\zeta$, $\beta = \sin^{-1}\mu$, the elliptic cylinder

$$\frac{x^2}{a^2\sinh^2\alpha_0}+\frac{y^2}{a^2\cosh^2\alpha_0} = 1$$

creates, in motion parallel to x, the potential

$$\phi = \frac{a\mathrm{U}\sin\beta(\cosh\alpha-\sinh\alpha)}{1-\tanh\alpha_0}$$

which is, of course, readily deducible also from the two-dimensional solution.

The elliptic anchor ring, the minor axis of whose section is parallel to the axis of the ring, produces, in motion along the axis of the ring, a velocity potential ϕ close to its surface, to which this formula gives a first approximation. The co-ordinates $(\alpha\beta)$ are those already used in this paper, and the present mode of approach indicates their relation to the usual "elliptic co-ordinates" of two dimensions.

Thus for motion of an elliptic anchor ring of the type in question,

$$\phi = \sin\beta e^{-\alpha}$$

is the solution of

$$\frac{\partial^2\phi}{\partial\alpha^2}+\frac{\partial^2\phi}{\partial\beta^2} = 0$$

to which the true velocity potential, satisfying

$$\frac{\partial^2\phi}{\partial\alpha^2}+\frac{\partial^2\phi}{\partial\beta^2} = -\frac{a}{c}\left\{\cos\beta\frac{\partial}{\partial\alpha}\cosh\alpha\frac{\partial\phi}{\partial\alpha}+\cosh\alpha\frac{\partial}{\partial\beta}\cos\beta\frac{\partial\phi}{\partial\beta}\right\},$$

approximates when a/c tends to zero.

Write therefore, for the anchor ring, or for the circular annulus $\alpha_0 = 0$ which we shall alone consider.

$$\phi = e^{-\alpha}\sin\beta+\frac{a}{c}\psi_1+\frac{a^2}{c^2}\psi_2+\frac{a^3}{c^3}\psi_3+\ldots$$

where ψ_1, ψ_2, ... ψ_n have the properties:—

(1) $\psi_n = 0$ when $\beta = 0$, over the rest of the plane $z = 0$,

(2) $\partial\psi_n/\partial\alpha = 0$ when $\alpha = 0$, over the annulus,

and all the conditions of our problem will be satisfied.

We find

$$\frac{\partial^2\psi_{n+1}}{\partial\alpha^2}+\frac{\partial^2\psi_{n+1}}{\partial\beta^2} = -\left\{\cos\beta\frac{\partial}{\partial\alpha}\cosh\alpha\frac{\partial}{\partial\alpha}+\cosh\alpha\frac{\partial}{\partial\beta}\cos\beta\frac{\partial}{\partial\beta}\right\}\psi_n,$$

where no complementary function satisfying $\frac{\partial^2\psi_{n+1}}{\partial\alpha^2}+\frac{\partial^2\psi_{n+1}}{\partial\beta^2} = 0$ must be added at any stage. It is understood that this development of the solution is only valid near the annular surface.

Proceeding to the successive approximations,

$$\frac{\partial^2\psi_1}{\partial\alpha^2}+\frac{\partial^2\psi_1}{\partial\beta^2} = -\left\{\cos\beta\frac{\partial}{\partial\alpha}\cosh\alpha\frac{\partial}{\partial\alpha}+\cosh\alpha\frac{\partial}{\partial\beta}\cos\beta\frac{\partial}{\partial\beta}\right\}e^{-\alpha}\sin\beta$$
$$= \tfrac{1}{2}\sin 2\beta,$$

whence $\psi_1 = -\frac{1}{8}\sin 2\beta$, and the second approximation to ϕ is

$$\phi = e^{-\alpha}\sin\beta-\frac{a}{8c}\sin 2\beta.$$

Again,

$$\frac{\partial^2\psi_2}{\partial\alpha^2}+\frac{\partial^2\psi_2}{\partial\beta^2} = +\tfrac{1}{8}\left\{\cos\beta\frac{\partial}{\partial\alpha}\cosh\alpha\frac{\partial}{\partial\alpha}+\cosh\alpha\frac{\partial}{\partial\beta}\cos\beta\frac{\partial}{\partial\beta}\right\}\sin 2\beta$$
$$= -\tfrac{1}{8}\cosh\alpha\,(\sin\beta+3\sin 3\beta),$$

whence $$\psi_2 = \tfrac{1}{64}\cosh\alpha\,(3\sin 3\beta+4\beta\cos\beta),$$

and the third approximation is

$$\phi = e^{-\alpha}\sin\beta-\frac{a}{8c}\sin 2\beta+\frac{a^2}{64c^2}\cosh\alpha\,(3\sin 3\beta+4\beta\cos\beta).$$

We do not proceed further, as the general term cannot be recognised. On the surface $\alpha = 0$,

$$\phi = \phi_0 = \sin\beta-\frac{a}{8c}\sin 2\beta+\frac{a^2}{64c^2}(3\sin 3\beta+4\beta\cos\beta)$$

$$-\left(\frac{\partial\phi}{\partial n}\right)_0 = \frac{1}{a}$$

from which the kinetic energy can be found. If σ is the density of the liquid, the whole kinetic energy of the liquid is T, where

$$2\mathrm{T} = -2\sigma\iint\phi_0\left(\frac{\partial\phi}{\partial n}\right)_0 d\mathrm{S}$$

taken over one side of the annulus. Thus if ϕ, $\partial\phi/\partial n$ are multiplied by $a\mathrm{U}$, where U is the velocity of the annulus,

$$\mathrm{T} = 2\pi\sigma a\mathrm{U}^2\int_{c-a}^{c+a}\rho\,d\rho\,\phi_0.$$

But on the surface,

$$\rho = c+a\sqrt{(1-\mu^2)} = c+a\cos\beta, \qquad d\rho = -a\sin\beta\,d\beta,$$

and $$\mathrm{T} = 2\pi\sigma a^2\mathrm{U}^2\int_0^{\pi}\sin\beta\,(c+a\cos\beta)\,\phi_0\,d\beta,$$

or $$\mathrm{T} = 2\pi\sigma a^2\mathrm{U}^2(\mathrm{T}_1+\mathrm{T}_2+\mathrm{T}_3+\mathrm{T}_4)$$

where

$$T_1 = \int_0^\pi \sin^2\beta\,(c+a\cos\beta)\,d\beta = \tfrac{1}{2}\pi c,$$

$$T_2 = -\frac{a}{4c}\int_0^\pi \sin^2\beta\cos\beta\,(c+a\cos\beta)\,d\beta = -\frac{\pi a^2}{32c},$$

$$T_3 = \frac{3a^2}{64c^2}\int_0^\pi (3\sin^2\beta-4\sin^4\beta)(c+a\cos\beta)\,d\beta = 0,$$

$$T_4 = \frac{a^2}{16c^2}\int_0^\pi \beta\sin\beta\cos\beta\,(c+a\cos\beta)\,d\beta$$

$$= \frac{a^2}{64c^2}\int_0^\pi \beta\,(2c\sin 2\beta+a\sin 3\beta+a\sin\beta)$$

$$= \frac{a^2}{64c^2}\Big[\beta\,(-c\cos 2\beta-\frac{a}{3}\cos 3\beta-a\cos\beta)\Big]_0^\pi + \frac{a^2}{64c^2}\int_0^\pi d\beta\,(c\cos 2\beta+\frac{a}{3}\cos 3\beta+a\cos\beta)$$

$$= -\frac{\pi a^2}{64c^2}\left(c-\frac{4a}{3}\right),$$

or to relative order a^2/c^2,

$$T = \pi^2\sigma a^2 c U^2\left(1-\frac{3}{32}\frac{a^2}{c^2}\right).$$

Series of zonal harmonics for the value of ϕ at any point can easily be obtained by the reader, making use of the method already adopted.

It is evident that the corresponding electrical problem, for which this value of ϕ holds, is that of an annular opening in an infinite plane conductor. The surface densities σ of the free distribution on the two portions of the conductor, when at the same potential, are given by $(ac)^{-1}(\partial\phi/\partial\beta)_{\beta=0}$, being infinite at each edge, or

$$a\zeta\sigma = \cosh\alpha-\sinh\alpha-\frac{a}{4c}+\frac{13}{64}\frac{a^2}{c^2}\cosh\alpha.$$

On the inner surface,

$$\cosh\alpha = \sqrt{(1+\zeta^2)} = \frac{c-\rho}{a},$$

$$\sinh\alpha = \zeta = \frac{1}{a}\sqrt{(c-a-\rho)(c-a+\rho)},$$

and on the outer,

$$\cosh\alpha = \sqrt{(1+\zeta^2)} = \frac{\rho-c}{a},\quad \sinh\alpha = \zeta = \frac{1}{a}\sqrt{\{(\rho-c-a)(\rho-c+a)\}}$$

and the surface densities can be written down. It is an interesting, though obvious, matter to work out the capacity of such an arrangement, by the method previously adopted for the capacity of an annulus.